

Closing Tues: HW 13.4, 14.1

Closing Thur: HW 14.2 (part 1)

14.1/14.2 Partial Derivatives

Goal: To find derivatives of multivariable functions.

Idea: Look at one variable at a time.

Entry Task: Consider

$$f(x, y) = 4xy + y^2 - 3x - 5y$$

1. Plug in $y = 1$, then find the derivative with respect to x .

2. Do it again with $y = 2$, ...

3. And again with $y = 3$, ...

4. Plug in $x = 1$, then find the derivative with respect to y .

5. Do it again with $x = 2$, ...

6. And again with $x = 3$, ...

$$\boxed{1} \quad f(x, 1) = 4x + 1 - 3x - 5 = x - 4$$
$$\frac{dz}{dx} = 4 + 0 - 3 - 0 = 1$$

$$\boxed{2} \quad f(x, 2) = 4x(2) + (2)^2 - 3x - 5(2)$$
$$= 8x + 4 - 3x - 10$$
$$\frac{dz}{dx} = 8 + 0 - 3 + 0 = 5$$

$$\boxed{3} \quad f(x, 3) = 4x(3) + (3)^2 - 3x - 5(3)$$
$$\frac{dz}{dx} = 12 + 0 - 3 - 0 = 9$$

$$\boxed{4} \quad f(1, y) = 4(1)y + y^2 - 3(1) - 5y$$
$$\frac{dz}{dy} = 4 + 2y - 0 - 5$$

$$\boxed{5} \quad f(2, y) = 4(2)y + y^2 - 3(2) - 5y$$
$$\frac{dz}{dy} = 8 + 2y - 0 - 5$$

$$\boxed{6} \quad f(3, y) = 4(3)y + y^2 - 3(3) - 5y$$
$$\frac{dz}{dy} = 12 + 2y - 0 - 5$$

Recall: Definition of derivative

1. Given a function $y = f(x)$.
2. Simplify the general formula for the slope of the secant from x to $x + h$

$$\frac{f(x+h) - f(x)}{h}$$

3. Let $h \rightarrow 0$, to get

$$\frac{dy}{dx} = f'(x) = \text{slope of tangent}$$

Partial Derivatives

For multivariable functions:

1. Given $z = f(x, y)$
- 2a. Simplify (y fixed, x variable)

$$\frac{f(x+h, y) - f(x, y)}{h}$$

- 3a. Let $h \rightarrow 0$, to get

$$\frac{\partial z}{\partial x} = f_x(x, y) \quad (\text{with respect to } x)$$

Example:

$$f(x, y) = 4xy + y^2 - 3x - 5y$$

$$\frac{f(x+h, y) - f(x, y)}{h} =$$

$$\frac{[4(x+h)y + y^2 - 3(x+h) - 5y] - [4xy + y^2 - 3x - 5y]}{h}$$

$$\frac{4xy + 4hy + y^2 - 3x - 3h - 5y - 4xy - y^2 + 3x + 5y}{h}$$

$$= \frac{4hy - 3h}{h} = 4y - 3$$

$$\frac{\partial z}{\partial x} = f_x(x, y) = 4y - 3$$

x is variable!

$$\begin{array}{cccc} 4xy + y^2 - 3x - 5y & & & \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 4(1)y + 0 - 3 & & & -0 \\ 4y - 3 & & & \end{array}$$

2b. Simplify (x fixed, y variable)

$$\frac{f(x, y+h) - f(x, y)}{h}$$

3b. Let $h \rightarrow 0$, to get

$$\frac{\partial z}{\partial y} = f_y(x, y) \quad (\text{with respect to } x)$$

y is variable!

$$\frac{\partial z}{\partial y} = 4x + 2y - 5$$

$$4xy + y^2 - 3x - 5y$$

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$$4x(1) + 2y - 0 - 5$$

Example:

$$f(x, y) = 4xy + y^2 - 3x - 5y$$

$$\frac{f(x, y+h) - f(x, y)}{h} =$$

$$\frac{[4x(y+h) + (y+h)^2 - 3x - 5(y+h)] - [4xy + y^2 - 3x - 5y]}{h}$$

$$\frac{4xy + 4xh + y^2 + 2yh + h^2 - 3x - 5y - 5h - 4xy - y^2 + 3x + 5y}{h}$$

$$\frac{4xh + 2yh + h^2 - 5h}{h}$$

$$= 4x + 2y + h - 5 \quad h \rightarrow 0$$

How to do partial derivatives:

Step 0: Rewrite powers and simplify like we always do.

Step 1: Identify the desired variable!
(Underline it if it helps)
Treat all other variable like numbers!

Step 2: Identify the constants terms and the coefficients.
"Bring down coefficients"

Step 3: Use the regular one-variable derivative rules.

Example: Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for

$$1. z = 10x^4 + 7xy^3 + 8x^2y^{10}$$

$$\frac{\partial}{\partial x} : 10(4x^3) + 7(1)y^3 + 8(2x)y^{10}$$

$$\frac{\partial z}{\partial x} = 40x^3 + 7y^3 + 16xy^{10}$$

$$\frac{\partial}{\partial y} : 0 + 7x(3y^2) + 8x^2(10y^9)$$

$$\frac{\partial z}{\partial y} = 21xy^2 + 80x^2y^9$$

More examples: Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for

1. $z = x^3 - y^2 + 3xy^4$

$$\frac{d}{dx} : (3x^2) - 0 + 3(1)y^4$$

$$\boxed{\frac{\partial z}{\partial x} = 3x^2 + 3y^4}$$

$$\frac{d}{dy} : 0 - (2y) + 3x(4y^3)$$

$$\boxed{\frac{\partial z}{\partial y} = -2y + 12xy^3}$$

2. $z = e^{x^2} - \ln(y) + 7$

$$\frac{d}{dx} : (e^{x^2} \cdot 2x) - 0 + 0$$

$$\boxed{\frac{\partial z}{\partial x} = e^{x^2} \cdot (2x) = 2xe^{x^2}}$$

$$\frac{d}{dy} : 0 - \left(\frac{1}{y}\right) + 0$$

$$\boxed{\frac{\partial z}{\partial y} = -\frac{1}{y}}$$

$$3. z = (x^2 + 3y)^{10}$$

$$\frac{d}{dx} : 10(x^2 + 3y)^9 \cdot (2x + 0)$$

$$\boxed{\frac{dz}{dx} = 20x(x^2 + 3y)^9}$$

$$\frac{d}{dy} : 10(x^2 + 3y)^9 \cdot (0 + 3)$$

$$\boxed{\frac{dz}{dy} = 30(x^2 + 3y)^9}$$

$$4. z = \underbrace{xy^2}_{F} \underbrace{e^x}_{S}$$

$$\frac{d}{dx} : \text{product rule! } F = x \quad F' = 1$$
$$S = y^2 e^x$$
$$S' = y^2 (e^x)$$

$$x y^2 e^x + (1) y^2 e^x$$
$$F S' + F' S$$

$$\boxed{\frac{dz}{dx} = xy^2 e^x + y^2 e^x}$$

$$\frac{d}{dy} : x(2y) e^x$$

$$\boxed{\frac{dz}{dy} = 2xy e^x}$$

Interpreting as a rate

Your company produces and sells **two** products (hats and sunglasses)

x = number of hats

y = number of glasses

You find that profit is given by

$$P(x, y) = -3x^2 + 30x - 5y^2 + 130y + 2xy - 100$$

1. Find the partial derivatives.

2. Find and interpret

$P_x(5, 8)$ and $P_y(5, 8)$.

$$P_x = -6x + 30 + 2y$$

$$P_y = -10y + 130 + 2x$$

$$P_x(5, 8) = -6(5) + 30 + 2(8) = 16 = \frac{\partial P}{\partial x} \leftarrow \begin{array}{l} \text{dollar in profit per} \\ \text{hats} \end{array}$$

"The sale of the next hat will increase profit by about \$16." \uparrow after (5, 8)

$$P_y(5, 8) = -10(8) + 130 + 2(5) = -80 + 130 + 10 = 60 = \frac{\partial P}{\partial y} \leftarrow \begin{array}{l} \text{dollar in profit per} \\ \text{sunglasses} \end{array}$$

"The sale of the next sunglasses will increase profit by about \$60."

3. Estimate the values of

$$\frac{P(5.001, 8) - P(5, 8)}{0.001} \approx P_x(5, 8) = 16$$

$$\frac{P(5, 8.01) - P(5, 8)}{0.01} \approx P_y(5, 8) = 60$$

y -changing

Graphical Interpretation

Pretend you are skiing on the surface
 $z = f(x, y) = 15 - x^2 - y^2$

1. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

2. Find and interpret
 $f_x(7, 4)$ and $f_y(7, 4)$

$$f_x = -2x$$

$$f_y = -2y$$

$$f_x(7, 4) = -14$$

$$f_y(7, 4) = -8$$

Increase x slightly \Rightarrow output changes with slope = -14
"rate"

Increase y slightly \Rightarrow output changes with rate = -8